Light-front time picture of the Bethe-Salpeter equation fermionic

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The reduction of the equation of Bethe-Salpeter of two fermions in front of light is studied for the Yukawa model. We use the light-front Green's function for the N-particle system for two-fermions plus N-2 intermediate bosons.

1 Light-front

The light-front coordinates are defined in terms of these by the following relations:

$$x^+ = x^0 + x^3,\ x^- = x^0 - x^3 \text{ and } \vec{x}^\perp = x^1\vec{i} + x^2\vec{j}, \tag{1}$$

where $\vec{i}$ and $\vec{j}$ are the unit vectors in the direction of the coordinates x and y. The null plane is defined by $x^+ = 0$, that is, this condition defines the hyper-surface which is tangent to the light cone, the reason why some authors call those light-cone coordinates.

Note that for the usual four-dimensional Minkowski space-time whose metric $g^{\mu\nu}$ is defined such that its signature is $(1, -1, -1, -1)$ we have

$$\begin{aligned}
x^+ &= x^0 + x^3 = x_0 - x_3 \equiv x_-, \\
x^- &= x^0 - x^3 = x_0 + x_3 \equiv x_+, \\
\vec{x}^\perp &= x^1\vec{i} + x^2\vec{j} = -x_1\vec{i} - x_2\vec{j} \equiv -x_\perp,
\end{aligned} \tag{2}$$

The initial boundary conditions for the dynamics in the light front are defined in this hyper-plane. Note that the axis x^+ is orthogonal to the plane $x^+ = 0$. Therefore, a displacement of this hyper-surface for $x^+ > 0$ is analogous to the displacement of the plane $t = 0$ for $t > 0$ of the usual four-dimensional space-time. With this analogy we identify x^+ as the "time" coordinate for the null plane. Of course, since there is a conspicuous discrete symmetry between $x^+ \leftrightarrow x^-$, one could choose x^- as his "time" coordinate. However, once chosen, one has to stick to the convention adopted. We shall adhere to the former one.

The canonically conjugate momenta for the coordinates x^+, x^- and $x^\perp$ are defined respectively by:

$$k^+ = k^0 + k^3,\ k^- = k^0 - k^3 \text{ and } k^\perp = \left(k^1, k^2\right). \tag{3}$$

The scalar product in the light front coordinates becomes therefore

$$a^\mu b_\mu = \frac{1}{2}\left(a^+b^- + a^-b^+\right) - \vec{a}^\perp \cdot \vec{b}^\perp, \tag{4}$$

where $\vec{a}^\perp$ and $\vec{b}^\perp$ are the transverse components of the four vectors. All four vectors, tensors and other entities bearing space-time indices such as Dirac matrices γ^μ can be expressed in this new way, using components $(+, -, \perp)$.

From (4) we can get the scalar product $x^\mu k_\mu$ in the light front coordinates as $x^\mu k_\mu = \frac{1}{2}\left(x^+k^- + x^-k^+\right) - \vec{x}^\perp \cdot \vec{k}^\perp$.

Here again, in analogy to the usual four-dimensional Minkowski space-time where such a scalar product is $x^\mu k_\mu = x^0k^0 - \mathbf{x} \cdot \mathbf{k}$ where $\mathbf{x}$ is the three-dimensional vector, with the energy k^0 associated to the time

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coordinate x^0, we have the light-front "energy" k^- associated to the light-front "time" x^+. Note, however, that there is a crucial difference between the two formulations: while the usual four-dimensional space-time is Minkowskian, the light-front coordinates projects this onto two sectorized Euclidean spaces, namely $(+,-)$, and $(\perp,\perp)$.

In the Minkowski space described by the usual space-time coordinates we have the relation between the rest mass and the energy for the free particle given by $k^\mu k_\mu = m^2$. Using (4), we have $k^\mu k_\mu = \frac{1}{2}\left(k^+k^- + k^-k^+\right) - \vec{k}^\perp \cdot \vec{k}^\perp$, so that

$$k^- = \frac{\overrightarrow{k}_\perp^2 + m^2}{k^+}. \tag{5}$$

Note that the energy of a free particle is given by $k^0 = \pm\sqrt{m^2 + \mathbf{k^2}}$, which shows us a quadratic dependence of k^0 with respect to $\mathbf{k}$. These positive/negative energy possibilities for such a relation were the source of much difficulty in the interpretation of the negative energy particle states in the beginning of the quantum field theory descriprion for particles, finally solved by the antiparticle interpretation given by Feynman. In contrast to this, we have a linear dependence between $(k^+)^{-1}$ and k^- (see Eq.(5)), which immediately reminds us of the non-relativistic quantum mechanical type of relationship for one particle state systems.

2 Boson and fermion propagator

The 1-body Green's functions can be derived from the covariant propagator for 1-particles propagating at equal light-front times. In this case the propagator from $x^+ = 0$ to $x^+ > 0$ is given by

$$S(x^+) = \frac{1}{2}\int \frac{dk^- dk^+ dk^\perp}{(2\pi)} \frac{i e^{\frac{-i}{2}k^- x^+}}{k^+\left(k^- - \frac{k_\perp^2 + m^2}{k^+} + \frac{i\varepsilon}{k^+}\right)}. \tag{6}$$

The Fourier transform to the total light-front energy (P^-) is given by $S(P^-) = \frac{1}{2}\int dx^+ e^{\frac{i}{2}P^- x^+} S(x^+)$ and the free 1-body Green's function is given by $S(k^-) = \frac{1}{k^+}G(k^-)$, where

$$G_0^{(1)}(k^-) = \frac{\theta(k^+)}{k^- - k^-_{on}} \tag{7}$$

with $k^-_{on} = \frac{k_\perp^2 + m^2}{k^+}$ being the light-front Hamiltonian of the free 1-particle system.

Let S_{F} denote fermion field propagator in covariant theory

$$S_{\mathrm{F}}(x^\mu) = \int \frac{d^4k}{(2\pi)^4} \frac{i(\not{k}_{\mathrm{on}} + m)}{k^2 - m^2 + i\varepsilon} e^{-ik^\mu x_\mu}, \tag{8}$$

where $\not{k}_{\mathrm{on}} = \frac{1}{2}\gamma^+ \frac{(k^\perp)^2 + m^2}{k^{+2}} + \frac{1}{2}\gamma^- k^+ - \gamma^\perp k^\perp$. Using light-front variables in the Eq.(8), we have

$$S_{\mathrm{F}}(x^+) = \frac{i}{2}\int \frac{dk^- dk^+ dk^\perp}{(2\pi)} \left[\frac{\not{k}_{on} + m}{k^+\left(k^- - k^-_{on} + \frac{i\varepsilon}{k^+}\right)} + \frac{\gamma^+}{2k^+}\right] e^{\frac{-i}{2}k^- x^+}. \tag{9}$$

We note that for the fermion field, light-front propagator differs from the Feynmam propagator by an instantaneous propagator.

The free 1-fermion Green's function is given by

$$G(k^-) = \frac{\Lambda_+(k)}{\left(k^- - k^-_{on} + \frac{i\varepsilon}{k^+}\right)}, \tag{10}$$

where $\Lambda_\pm(k) = \frac{\pm\not{k}_{on} + m}{2m}\theta(\pm k^+)$.

3 Coupled equations for the Green's functions

The light-front Green's function for the two fermions system obtained from the solution of the covariant BS equation that contains all two-body irreducible diagrams, with the exception of those including closed loops of bosons Ψ_1 and Ψ_2 and part of the cross-ladder diagrams, is given by:

$$G^{(2)}(K^-) = G_0^{(2)}(K^-) + G_0^{(2)}(K^-)VG^{(3)}(K^-)VG^{(2)}(K^-) \ , \tag{11}$$

$$G^{(3)}(K^-) = G_0^{(3)}(K^-) + G_0^{(3)}(K^-)VG_0^{(4)}(K^-)VG^{(3)}(K^-). \tag{12}$$

In the Yukawa model for fermions, the interaction operator acting between Fock-states differing by zero, one and two σ's, has matrix elements given by

$$\langle (q,s')k_\sigma|V|(k,s)\rangle = -2m(2\pi)^3\delta(q+k_\sigma-k)\frac{g_S}{\sqrt{q^+k_\sigma^+k^+}}\theta(k_\sigma^+)\overline{u}(q,s')u(k,s) \ , \tag{13}$$

$$\langle (q,s')k'_\sigma|V|(k,s)k_\sigma\rangle = -2(2\pi)^3\delta(q+k'_\sigma-k-k_\sigma)\delta_{s's}\frac{g_S^2}{\sqrt{k'^+_\sigma k_\sigma^+}}\frac{\theta(k'^+_\sigma)\theta(k^+_\sigma)}{k^++k^+_\sigma} \ , \tag{14}$$

$$\langle (q,s')k'_\sigma k_\sigma|V|(k,s)\rangle = -2(2\pi)^3\delta(q+k'_\sigma+k_\sigma-k)\delta_{s's}\frac{g_S^2}{\sqrt{k'^+_\sigma k_\sigma^+}}\frac{\theta(k'^+_\sigma)\theta(k^+_\sigma)}{k^+-k^+_\sigma} \ . \tag{15}$$

The instantaneous terms in the two-fermion propagator give origin to Eqs. (14) and (15).

A systematic expansion by the consistent truncation of the light-front Fock space up to N particles in the intermediate states (boson 1, boson 2 and $N-2$ σ's) in the set of Eqs.(12) and (11), amounts to substitution $G^{(3)}(K^-) \cong G_0^{(3)}(K^-)$. The kernel of Eq.(11) still contains an infinite sum of light-front diagrams, that are obtained solving by Eq.(12). To obtain the ladder aproximation up to order g^2, Eq.(11), only the free and first order terms are kept in Eq.(12), with the restriction of only one and one boson covariant exchanges.Therefore, we have for Eq.(11)

$$G_{g^2}^{(2)}(K^-) = G_0^{(2)}(K^-) + G_0^{(2)}(K^-)VG_0^{(3)}(K^-)VG_{g^2}^{(2)}(K^-), \tag{16}$$

Taking the two-boson system as an example and restricting the intermediate state propagation up to 3-particles, we find that

$$G_{g^2}^{(2)}(K^-) = G_0^{(2)}(K^-) + G_0^{(2)}(K^-)VG_0^{(3)}(K^-)V\left\{G_0^{(2)}(K^-) + G_0^{(2)}(K^-)VG_0^{(3)}(K^-)VG_{g^2}^{(2)}(K^-)\right\} \tag{17}$$

The correction in order g^2 is given for $\Delta G_{g^2}^{(2)}(K^-) = G_0^{(2)}(K^-)VG_0^{(3)}(K^-)VG_0^{(2)}(K^-)$.

4 Bethe-Salpeter equation

We perform the quasi-potential reduction of two-body BSE's and present the coupled set of equations for the light-front Green's functions for bosonic and fermionic models, with the interaction Lagrangian respectively given by, $\mathcal{L}_I^B = g_S\phi_1^\dagger\phi_1\sigma + g_S\phi_2^\dagger\phi_2\sigma$ and $\mathcal{L}_I^F = g_S\overline{\Psi}\Psi\sigma$, where ϕ_1, ϕ_2 and σ are the bosonic fields and Ψ is the fermion field in the Yukawa model.

Close the area of energy of the bound state the Green's function has a pole $\lim_{K^-\to K_B^-} G^{(2)}(K^-) = \frac{|\psi_B\rangle\langle\psi_B|}{K^- - K_B^-}$, where $|\psi_B\rangle$ it is the wave-function of the bound state.

The homogeneous equation for the light-front two-body bound state wave-function is obtained the solution of

$$|\Psi_B >= G_0^{(2)}(K_B^-)VG^{(3)}(K_B^-)V|\Psi_B >, \tag{18}$$

The vertex function for the bound state wave-function is defined as

$$\Gamma_B(k_\perp, q^+) =< k,\ K-k| \left(G_0^{(2)}(K_B^-)\right)^{-1} |\Psi_B> . \tag{19}$$

The Green's function obtained from this equation, up to order g^2, reproduces the covariant two-body propagator between two light-front hypersurfaces. In this approximation, the vertex function satisfies the following integral equation,

$$\Gamma_B(\overrightarrow{q}_\perp, y) = \int \frac{dx d^2 k_\perp}{x(1-x)} \frac{K^{(3)}(\overrightarrow{q}_\perp, y; \overrightarrow{k}_\perp, x)}{M_B^2 - M_0^2} \Gamma_B(\overrightarrow{k}_\perp, x), \tag{20}$$

where the momentum fractions are $y = q^+/K^+$ and $x = k^+/K^+$, with $0 < y < 1$. Where $\overrightarrow{K}_\perp = 0$, and $M_0^2 = K^+ K^-_{(2)on} - K_\perp^2 = \frac{k_\perp^2 + m^2}{x(1-x)}$. The part of the kernel which contains only the propagation of virtual three particle states foward in the light-front time is obtained from Eq.(20) as,

$$\begin{aligned}\mathcal{K}^{(3)}(y, q_\perp; x, k_\perp) \quad &= \quad \frac{g^2}{16\pi^3} \frac{\Lambda_{+(1)}(q)\Lambda_{+(1)}(k)\Lambda_{+(2)}(K-q)\Lambda_{+(2)}(K-k)\theta(y-x)}{(x-y)\left(M_B^2 - \frac{\overrightarrow{q}_\perp^2 + m^2}{1-y} - \frac{\vec{k}_\perp^2 + m^2}{x} - \frac{(\overrightarrow{q}_\perp - \vec{k}_\perp)^2 + \mu^2}{y-x}\right)} + \\ &\quad + [k \leftrightarrow q] \,,\end{aligned}$$

being $M_B^2 = K_B^+ K_B^-$. W got the attention for the relationship between $|\Gamma_B\rangle$ and $\Gamma_B(\overrightarrow{q}_\perp, y)$, defined for $\Gamma_B(\overrightarrow{q}_\perp, y) = \sqrt{q^+(K^+ - q^+)}\,\langle \overrightarrow{q}_\perp, q^+\ |\Gamma_B\rangle$.

The generalization for any order g^n was calculated by the author [2].

5 Conclusion

We formulated the Bethe-Salpeter equation in the ladder aproximation on the light-front. The ladder comes from the exchange of an intermediate boson between the other two fermions. The kernel of the integral equation is constructed from the perturbative light-front propagator of the interacting two fermions system up to $\mathcal{O}(g^2)$.

We can use that method for the study of coupled virtual gauge boson [3].

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